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Solving high Reynolds-number viscous flows by the general BEM and domain decomposition method

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SUMMARY

In this paper, the domain decomposition method (DDM) and the general boundary element method (GBEM) are applied to solve the laminar viscous flow in a driven square cavity, governed by the exact Navier–Stokes equations. The convergent numerical results at high Reynolds number $Re = 7500$ are obtained. We find that the DDM can considerably improve the efficiency of the GBEM, and that the combination of the domain decomposition techniques and the parallel computation can further greatly improve the efficiency of the GBEM. This verifies the great potential of the GBEM for strongly non-linear problems in science and engineering. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: general boundary element method; domain decomposition method; parallel computing; Navier–Stokes equations; driven cavity flow

1. INTRODUCTION

To overcome the limitations of the traditional boundary element method (BEM) and widen its application in science and engineering, Liao and his co-authors [1–7] proposed the so-called general boundary element method (GBEM). The GBEM has a solid mathematical base, i.e. the homotopy analysis method [8–12]. In the past few years, the GBEM has been proved to be a powerful numerical technique for non-linear differential equations, especially for strongly non-linear problems.

However, domain integral is a challenge for the GBEM. It is well known that for non-linear problems the BEM loses its boundary-only merits due to the appearance of domain integrals, and thus is numerically inefficient in comparison with domain schemes such as the finite

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difference method (FDM) and the finite element method (FEM). Especially, it is unknown if the GBEM with domain integrals are efficient enough for three-dimensional non-linear problems.

To conquer the limitation of numerical inefficiency due to domain integrals, Zhao and Liao [7] successfully applied the parallel computation to the GBEM. For laminar viscous flows in a driven square cavity governed by the exact Navier–Stokes equations, they obtained the convergent numerical results at the high Reynolds number $Re = 7500$ by a high-performance parallel computer. This is the first time, to the best of our knowledge, to present convergent results at such a high Reynolds number by means of the BEM, which verifies the validity and great potential of the GBEM for strongly non-linear problems. However, when $Re = 7500$, a lot of CPU time is needed. Thus, it is necessary to further develop more efficient techniques for domain integrals in the GBEM.

Domain decomposition methods are developed long before the term 'domain decomposition' came into use [13, 14]. The popularity of the DDM [15–17] is motivated primarily either by the natural domain structure of the problems, such as non-homogeneous body and long body, or by the need to parallelize algorithms for large problems. Recently, Hribersek and Skeret [18] applied the DDM to the boundary element method of the Navier–Stokes equations. Kamiya *et al*. [19] proposed an algorithm for the parallel boundary element computation of the domain decomposed problem. Popov and Power [20, 21], Power and Mingo [22] applied the DDM to improve the accuracy of dual reciprocity method (DRM) and it turned out to be more efficient than the ordinary BEM.

In this paper, we apply the DDM to improve the numerical efficiency of the GBEM. By means of the DDM, the global domain is divided into several subdomains, and in each of them the full integral representation formula is applied. At the common interface between adjacent subregions, the corresponding full matching conditions, considering the compatibility and equilibrium conditions, are enforced. Each integral is restricted in the subdomain. As mentioned by Zhao and Liao [7], more than 95% CPU time of the GBEM approach is spent in domain integrals. Hence, the employment of the DDM can considerably improve the numerical efficiency of the GBEM.

Domain decomposition methods are naturally suitable for the parallel computation [19, 23–25]. To make our formulation numerically more efficient, we combine the DDM with the parallel computation. In this way, the efficiency of the GBEM is further improved greatly. Our numerical experiments show that, employing equal number of processors, the domain decomposition parallel computation is much more efficient than the pure parallel computation mentioned in Reference [7].

2. GENERAL BOUNDARY ELEMENT METHOD

Consider the two-dimensional steady viscous flow of an incompressible Newtonian fluid, governed in terms of the streamfunction ψ and the vorticity ω by

$$\nabla^2\omega = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right) \tag{1}$$

$$\nabla^2\psi + \omega = 0 \tag{2}$$

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subject to the boundary conditions

$$\psi = \psi_{\mathrm{b}} \qquad \text{on } \Gamma \tag{3}$$

$$\frac{\partial \psi}{\partial n} = \left(\frac{\partial \psi}{\partial n}\right)_{\mathrm{b}} \quad \text{on } \Gamma \tag{4}$$

where Re is the Reynolds number and Γ denotes the boundary of the global domain.

Using the homotopy analysis method, Liao [1] replaced the original non-linear equations (1)–(4) by a series of linear sub-problems, governed by

$$\nabla^2 \bar{\omega}_k(x, y) = S_{k-1}(x, y) \tag{5}$$

$$\nabla^2 \bar{\psi}_k(x, y) + \bar{\omega}_k(x, y) = 0 \tag{6}$$

subject to the related boundary conditions

$$\bar{\psi}_k(x, y) = (1 - \chi_k)\psi_{\mathrm{b}} \qquad \text{on } \Gamma \tag{7}$$

$$\frac{\partial \bar{\psi}_k(x, y)}{\partial n} = (1 - \chi_k)\left(\frac{\partial \psi}{\partial n}\right)_{\mathrm{b}} \quad \text{on } \Gamma \tag{8}$$

where

$$S_{k-1}(x, y) = Re \sum_{n=0}^{k-1} \left[\frac{\partial \psi_n}{\partial y}\frac{\partial \omega_{k-1-n}}{\partial x} - \frac{\partial \psi_n}{\partial x}\frac{\partial \omega_{k-1-n}}{\partial y}\right] \tag{9}$$

and

$$\chi_k = \begin{cases} 0 & \text{when } k \leqslant 1 \\ 1 & \text{when } k > 1 \end{cases} \tag{10}$$

Here $\bar{\omega}_k(x, y)$ and $\bar{\psi}_k(x, y)$ are defined by

$$\bar{\omega}_k(x, y) = \omega_k - (\chi_k - 1)\omega_{k-1} \tag{11}$$

$$\bar{\psi}_k(x, y) = \psi_k - (\chi_k - 1)\psi_{k-1} \tag{12}$$

Equations (5) and (6) with boundary conditions (7) and (8) can be solved by the traditional BEM. As soon as $\bar{\omega}_k(x, y)$, $\bar{\psi}_k(x, y)$ are known, we have via Equations (11) and (12) that

$$\omega_k = \bar{\omega}_k(x, y) + (\chi_k - 1)\omega_{k-1} \tag{13}$$

$$\psi_k = \bar{\psi}_k(x, y) + (\chi_k - 1)\psi_{k-1} \tag{14}$$

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The mth-order iterative formulations are as follows:

$$\omega_0^{i+1}(x,y) \leftarrow \omega_0^i(x,y) + \sum_{k=1}^{m} \omega_k^i(x,y)\, p^k \tag{15}$$

$$\psi_0^{i+1}(x,y) \leftarrow \psi_0^i(x,y) + \sum_{k=1}^{m} \psi_k^i(x,y)\, p^k \tag{16}$$

where the superscript i denotes the number of iteration and p is an embedding variable of homotopy. For details, please refer to Reference [1].

3. DOMAIN DECOMPOSITION METHOD

By means of the DDM, the whole domain Ω is decomposed into M subdomains $\Omega_1, \Omega_2, \ldots, \Omega_M$. In each subdomain, a full integral representation formula is applied. In the subdomain Ω_I $(I = 1, 2, \ldots, M)$, the boundary integral equations of Equations (5)–(8) can be written as

$$-a(\xi)\bar{\omega}_k^I(\xi) + \oint_{\Gamma_I} \left(G_\omega \frac{\partial \bar{\omega}_k^I}{\partial n} - \bar{\omega}_k^I \frac{\partial G_\omega}{\partial n} \right) \mathrm{d}\Gamma_I = \int_{\Omega_I} S_{k-1}^I G_\omega \, \mathrm{d}\Omega_I \tag{17}$$

$$-a(\xi)\bar{\psi}_k^I(\xi) + \oint_{\Gamma_I} \left(F_\psi \frac{\partial \bar{\omega}_k^I}{\partial n} - \bar{\omega}_k^I \frac{\partial F_\psi}{\partial n} \right) \mathrm{d}\Gamma_I = \int_{\Omega_I} S_{k-1}^I F_\psi \, \mathrm{d}\Omega_I - \oint_{\Gamma_I} \left(G_\omega \frac{\partial \bar{\psi}_k^I}{\partial n} - \bar{\psi}_k^I \frac{\partial G_\omega}{\partial n} \right) \mathrm{d}\Gamma_I \tag{18}$$

where the superscript I denotes the subdomain Ω_I, Γ_I the boundary of Ω_I, respectively. The parameter $a(\xi)$ is a geometric factor depending on the location of ξ:

$$a(\xi) = \begin{cases} 1 & \text{if } \xi \in \Omega_I \\ 0 & \text{if } \xi \in \Omega_I^c \\ \theta/2\pi & \text{if } \xi \in \Gamma_I \end{cases} \tag{19}$$

where Ω_I^c denotes the exterior of the domain Ω_I, excluding its boundary Γ_I, θ is the angle formed between the tangents to the boundary at point ξ, approaching it from each side. For points at which the boundary is differentiable, $\theta = \pi$, G_ω and G_ψ are fundamental solutions. According to Reference [26], we have

$$G_\omega = -\frac{1}{2\pi} \ln r \tag{20}$$

$$F_\psi = \frac{r^2}{8\pi}(\ln r - 1) \tag{21}$$

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Let Γ_I^{V} denote the interface of the subdomain Ω_I with other subdomains, and Γ_I^{R} the remainder boundary of the subdomain Ω_I, i.e. $\Gamma_I^{\mathrm{R}}+\Gamma_I^{\mathrm{V}}=\Gamma_I$. Obviously, Γ_I^{V} is a virtual boundary inner the global domain, on which the values of $\bar{\omega}_k^I$, $\partial\bar{\omega}_k^I/\partial n$, $\bar{\psi}_k^I$ and $\partial\bar{\psi}_k^I/\partial n$ are unknown. Note that, Γ_I^{R} is a real boundary, on which both ψ_k^I and $\partial\psi_k^I/\partial n$ are known, subject to the boundary conditions (7) and (8). According to Equation (19), if $\xi\in\Omega_I^c$ then $a(\xi)=0$. Thus, for points in the exterior of the domain Ω_I, excluding the boundary Γ_I, we have by substituting boundary conditions (7) and (8) into Equations (17) and (18) that

$$\oint_{\Gamma_I} G_\omega \frac{\partial\bar{\omega}_k^I}{\partial n}\,\mathrm{d}\Gamma_I - \oint_{\Gamma_I} \bar{\omega}_k^I \frac{\partial G_\omega}{\partial n}\,\mathrm{d}\Gamma_I = \int_{\Omega_I} S_{k-1}^I G_\omega\,\mathrm{d}\Omega_I \tag{22}$$

$$\begin{aligned}&\oint_{\Gamma_I} F_\psi \frac{\partial\bar{\omega}_k^I}{\partial n}\,\mathrm{d}\Gamma_I - \oint_{\Gamma_I} \bar{\omega}_k^I \frac{\partial F_\psi}{\partial n}\,\mathrm{d}\Gamma_I + \int_{\Gamma_I^{\mathrm{V}}} G_\omega \frac{\partial\bar{\psi}_k^I}{\partial n}\,\mathrm{d}\Gamma_I^{\mathrm{V}} - \int_{\Gamma_I^{\mathrm{V}}} \bar{\psi}_k^I \frac{\partial G_\omega}{\partial n}\,\mathrm{d}\Gamma_I^{\mathrm{V}}\\ &\quad= \int_{\Omega_I} S_{k-1}^I F_\psi\,\mathrm{d}\Omega_I - (1-\chi_k)\int_{\Gamma_I^{\mathrm{R}}}\left[G_\omega\left(\frac{\partial\bar{\psi}}{\partial n}\right)_{\mathrm{b}} - \psi_{\mathrm{b}}\frac{\partial G_\omega}{\partial n}\right]\mathrm{d}\Gamma_I^{\mathrm{R}}\end{aligned} \tag{23}$$

After discretizing Equations (22) and (23) by the boundary elements, the following matrix system is obtained:

$$\mathbf{G}_{(1)}^I\mathbf{w}_{\mathbf{n}}^I + \mathbf{H}_{(1)}^I\mathbf{w}^I = \mathbf{c}_{(1)}^I \tag{24}$$

$$\mathbf{F}_{(2)}^I\mathbf{w}_{\mathbf{n}}^I + \mathbf{E}_{(2)}^I\mathbf{w}^I + \mathbf{G}_{(2)}^I\mathbf{y}_{\mathbf{n}}^I + \mathbf{H}_{(2)}^I\mathbf{y}^I = \mathbf{c}_{(2)}^I \tag{25}$$

where $\mathbf{G}_{(1)}^I$, $\mathbf{H}_{(1)}^I$ denote the coefficient matrix determined from the first and the second integral of the left-hand side of Equation (22), $\mathbf{F}_{(2)}^I$, $\mathbf{E}_{(2)}^I$, $\mathbf{G}_{(2)}^I$, and $\mathbf{H}_{(2)}^I$ denote the coefficient matrix determined from the first, the second, the third, and the fourth integral of the left-hand side of Equation (23), $\mathbf{w}_{\mathbf{n}}^I$ and $\mathbf{w}^I$ denote the unknown vector of $\partial\bar{\omega}_k^I/\partial n$ and $\bar{\omega}_k^I$ on the boundary Γ_I of the subdomain Ω_I, $\mathbf{y}_{\mathbf{n}}^I$ and $\mathbf{y}^I$ denote the unknown vector of $\partial\bar{\psi}_k^I/\partial n$ and $\bar{\psi}_k^I$ on the boundary Γ_I^{V} of the subdomain Ω_I, and $\mathbf{c}_{(1)}^I$ and $\mathbf{c}_{(2)}^I$ denote the known vector determined from the right-hand side of Equations (22) and (23), respectively.

We can write the set of Equations (24) and (25) as

$$\begin{bmatrix}\mathbf{G}_{(1)}^I & \mathbf{H}_{(1)}^I & 0 & 0\\ \mathbf{F}_{(2)}^I & \mathbf{E}_{(2)}^I & \mathbf{G}_{(2)}^I & \mathbf{H}_{(2)}^I\end{bmatrix}\begin{bmatrix}\mathbf{w}_{\mathbf{n}}^I\\ \mathbf{w}^I\\ \mathbf{y}_{\mathbf{n}}^I\\ \mathbf{y}^I\end{bmatrix} = \begin{bmatrix}\mathbf{c}_{(1)}^I\\ \mathbf{c}_{(2)}^I\end{bmatrix} \tag{26}$$

i.e. in the subdomain Ω_I, we have a system of algebraic equations of the form

$$\mathbf{B}^I\mathbf{x}^I = \mathbf{c}^I, \quad I=1,2,\ldots,M \tag{27}$$

where the superscript I denotes the subdomain Ω_I, M is the number of subdomains, and

$$\mathbf{B}^I = \begin{bmatrix}\mathbf{G}_{(1)}^I & \mathbf{H}_{(1)}^I & 0 & 0\\ \mathbf{F}_{(2)}^I & \mathbf{E}_{(2)}^I & \mathbf{G}_{(2)}^I & \mathbf{H}_{(2)}^I\end{bmatrix} \tag{28}$$

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is a matrix which depends only on the geometry of the boundary.

$$\mathbf{x}^I = [\mathbf{w}_\mathbf{n}^I \;\; \mathbf{w}^I \;\; \mathbf{y}_\mathbf{n}^I \;\; \mathbf{y}^I]^\mathrm{T} \tag{29}$$

is a vector of the unknowns on the boundary of the subdomains Ω_I, and

$$\mathbf{c}^I = [\mathbf{c}_{(1)}^I \;\; \mathbf{c}_{(2)}^I]^\mathrm{T} \tag{30}$$

is a vector of known values.

On the common interface Γ_{IJ}^V of the subdomains Ω_I and Ω_J, we use the matching conditions:

$$(\mathbf{w})_{\Gamma_{IJ}^\mathrm{V}}^I = (\mathbf{w})_{\Gamma_{IJ}^\mathrm{V}}^J \tag{31}$$

$$(\mathbf{y})_{\Gamma_{IJ}^\mathrm{V}}^I = (\mathbf{y})_{\Gamma_{IJ}^\mathrm{V}}^J \tag{32}$$

$$(\mathbf{w}_\mathbf{n})_{\Gamma_{IJ}^\mathrm{V}}^I = -(\mathbf{w}_\mathbf{n})_{\Gamma_{IJ}^\mathrm{V}}^J \tag{33}$$

$$(\mathbf{y}_\mathbf{n})_{\Gamma_{IJ}^\mathrm{V}}^I = -(\mathbf{y}_\mathbf{n})_{\Gamma_{IJ}^\mathrm{V}}^J \tag{34}$$

Each of the M local matrix systems given by Equation (27) can be assembled with its neighbouring systems according to the matching conditions given by Equations (31)–(34). The assembled global systems can be written in the following form:

$$\mathbf{B}\mathbf{x} = \mathbf{c} \tag{35}$$

where $\mathbf{B}$ is an assembled spare global coefficient matrix, $\mathbf{x}$ is an assembled global unknown vector in the boundaries of each subdomain, and $\mathbf{c}$ is an assembled global known vector. Solving the linear equation (35), we get $\bar{\omega}_k$, $\partial\bar{\omega}_k/\partial n$, $\bar{\psi}_k$, and $\partial\bar{\psi}_k/\partial n$ on the boundary of each subdomain. One of the advantages of the GBEM is that the global coefficient matrix $\mathbf{B}$ maintains the same in solving $\bar{\omega}_k$, $\partial\bar{\omega}_k/\partial n$, $\bar{\psi}_k$, and $\partial\bar{\psi}_k/\partial n$ for any $k \geqslant 1$ in each iteration. Therefore, instead of solving such a spare linear system by direct or iterative solvers, we need only invert this matrix by the LU decomposition one time, and the inverse matrix can be stored and used again and again.

Once $\bar{\omega}_k^I$, $\partial\bar{\omega}_k^I/\partial n$, $\bar{\psi}_k^I$, and $\partial\bar{\psi}_k^I/\partial n$ on the boundary Γ_I of the subdomain Ω_I are known, using Equations (17)–(19), we can obtain $\bar{\omega}_k^I$, and $\bar{\psi}_k^I$ in the subdomain Ω_I:

$$\bar{\omega}_k^I(x,y) = \oint_{\Gamma_I} \left(G_\omega \frac{\partial\bar{\omega}_k^I}{\partial n} - \bar{\omega}_k^I \frac{\partial G_\omega}{\partial n} \right) \mathrm{d}\Gamma_I - \int_{\Omega_I} S_{k-1}^I \, G_\omega \, \mathrm{d}\Omega \tag{36}$$

$$\begin{aligned} \bar{\psi}_k^I(x,y) = {} & \oint_{\Gamma_I} \left(F_\psi \frac{\partial\bar{\omega}_k^I}{\partial n} - \bar{\omega}_k^I \frac{\partial F_\psi}{\partial n} \right) \mathrm{d}\Gamma_I + \int_{\Gamma_I^\mathrm{V}} \left(G_\omega \frac{\partial\bar{\psi}_k^I}{\partial n} - \bar{\psi}_k^I \frac{\partial G_\omega}{\partial n} \right) \mathrm{d}\Gamma_I^\mathrm{V} \\ & - \int_{\Omega_I} S_{k-1}^I \, F_\psi \, \mathrm{d}\Omega_I + (1-\chi_k) \int_{\Gamma_I^\mathrm{R}} \left[G_\omega \left(\frac{\partial\bar{\psi}}{\partial n} \right)_\mathrm{b} - \psi_\mathrm{b} \frac{\partial G_\omega}{\partial n} \right] \mathrm{d}\Gamma_I^\mathrm{R} \end{aligned} \tag{37}$$

The final solutions are obtained using (13)–(16).

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4. PARALLEL COMPUTING

As mentioned before, domain decomposition methods are naturally suitable for parallel computation. Actually, many domain decomposition methods are mainly designed for parallel implementation, especially in the finite element method. If each subdomain is assigned a processor, it is very convenient to parallelize our approach. This parallelization strategy is often employed in domain decomposition parallel computation [27].

In this paper, to make the GBEM numerically more efficient, we combine the DDM with the data decomposition parallelization strategy [28] in each subdomain. Instead of assigning only *one* processor to each subdomain in the pure domain decomposition parallel computation, we divide the total processors into M groups, and assign a *group* of processors to each subdomain. In each subdomain, data decomposition parallelization strategy is employed to calculate the domain integrals over this subdomains. The data decomposition parallelization strategy has been proved to be efficient for the GBEM by Zhao and Liao [7]. The difference lies in that our data decomposition schemes are executed within the subdomain while Zhao and Liao's [7] strategies are implemented in the global domain.

The message between processors is conveyed by message passing interface (MPI) [29]. Our parallel codes are implemented on a high-performance parallel computer SGI Onyx 3800, a supercomputer containing 64 processors (500 MHz) with 32 GB shared memory.

5. NUMERICAL EXAMPLE

For example, let us consider the viscous flow in a square cavity. The geometry and the corresponding boundary conditions are as shown in Figure 1. For convenience, the global domain is decomposed into four, nine, and sixteen uniform subdomains, respectively, as illustrated in Figure 1. The parallelization strategy combining domain decomposition and data decomposition is illustrated in Figure 2. Each subdomain has the same number of boundary elements and the same number of processors. We use uniform rectangle numerical grids. Like Liao [1], linear boundary elements are used. At the corners of each subdomain, a so-called 'double node approach', that is, using two nodes close to each other at the corner, one belonging to each side, is applied. Throughout this paper, we use the solution at $Re = 0$ as our initial approximation. The criterion for convergence is defined by $\varepsilon = |(f^{m+1} - f^m)/f^m| < 5 \times 10^{-3}$.

In this paper, the speedup is defined as follows:

$$\text{speedup} = \frac{t_0}{t_1} \tag{38}$$

where t_0 denotes the time needed in a single domain without parallel computation, and t_1 denotes the time needed with domain decomposition or parallel computation. From the physical view it denotes the speedup effect of our domain decomposition method, or the parallel computation, or the combination between these two methods.

To demonstrate the speedup effect of the domain decomposition method in the GBEM, we firstly implement our code in a personal computer, with only one processor of Intel Pentium 4 (1.5 GHz). The speedup given by different number of subdomains with different mesh is shown in Table I. In the case of 16 subdomains, it is obvious that the speedup is higher if finer mesh is used. This is mainly because when finer mesh is used, relatively more CPU

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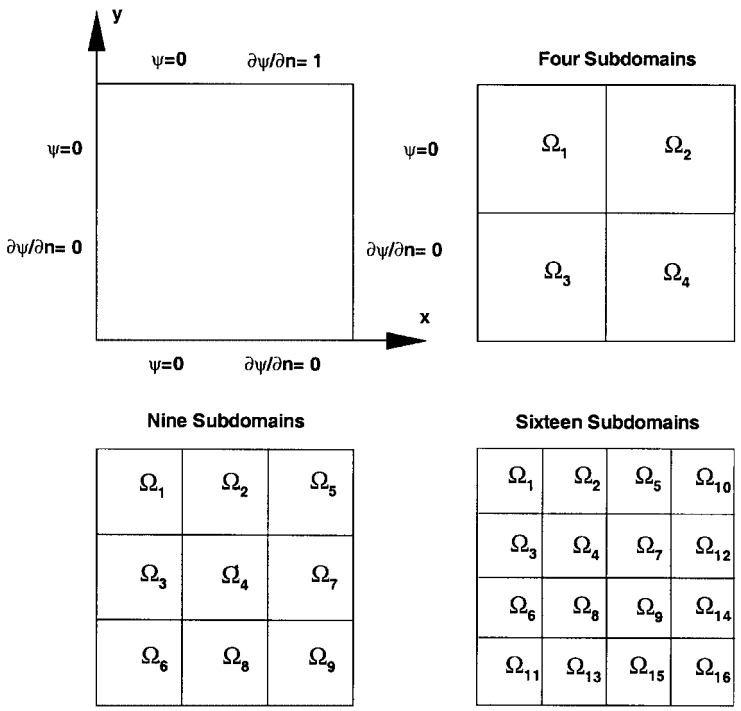

Figure 1. Geometry of the driven cavity viscous flow with boundary conditions and domain decompositions.

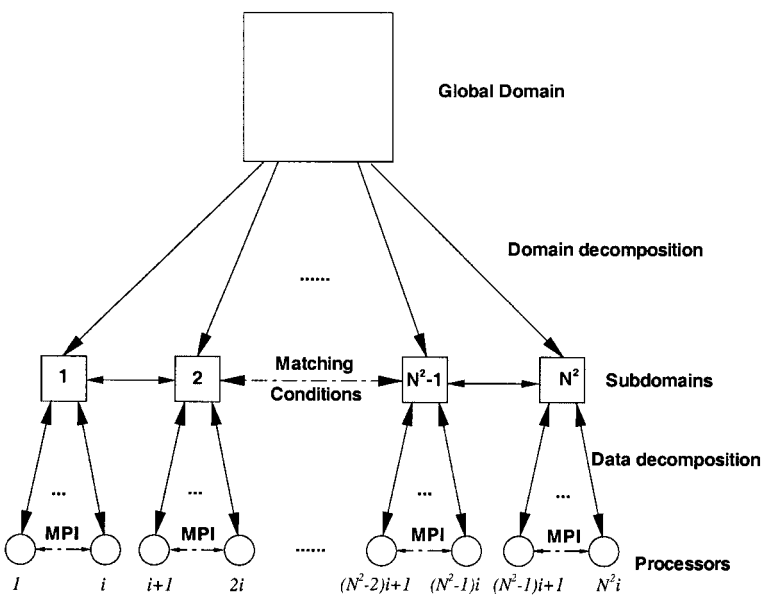

Figure 2. Parallel strategy combining domain decomposition and data decomposition.

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Table I. The speedup given by different number of subdomains.

Mesh	Four subdomains	Nine subdomains	Sixteen subdomains
48×48	3.74	8.60	10.12
72×72	3.76	8.19	12.05
96×96	3.88	8.72	13.15
120×120	3.83	8.24	13.74

Table II. The number of iteration and corresponding CPU time at different Reynolds number by global domain ($1D$), four subdomains ($4D$), nine subdomains ($9D$), and 16 subdomains ($16D$) on personal computer.

		Number of iteration				Relative CPU time			
Re	*p*	$1D$	$4D$	$9D$	$16D$	$1D$	$4D$	$9D$	$16D$
100	0.5	20	19	19	20	1	0.251	0.115	0.0815
400	0.2	73	76	81	74	3.62	0.976	0.467	0.285
1000	0.05	259	259	245	261	12.8	3.31	1.40	0.988
3200	0.01	1311	1403	1219	1471	154.6	43.3	17.5	12.6
5000	0.005	\	3039	3121	3081	\	93.7	44.7	26.4
7500	0.002	\	\	7111	6931	\	\	101.8	59.5

time are spent in domain integrals. The accurate enough solutions for $Re = 100$, 400, and 1000 are obtained by 96×96 mesh grid, and for $Re = 3200$, 5000, and 7500 by 120×120 mesh grid. The iterative times and corresponding relative CPU time to get convergent solutions by a personal computer are listed in Table II. Here we take the CPU time consumption for the solution at $Re = 100$ on 96×96 mesh grid as unity. It is clear that by means of the DDM, the total CPU time is greatly decreased while the number of iteration maintains nearly the same, as shown in Table II.

It should be pointed out that employing the GBEM and the domain decomposition method, we can get the convergent solutions even for the Reynolds number $Re = 10\,000$. As mentioned by Zhao and Liao [7], using the GBEM, we can always find a small enough p to ensure that the iteration is convergent at high Reynolds numbers. Therefore, our numerical method is valid in the whole range of Reynolds numbers. However, without the DDM, it is numerically inefficient to obtain the convergent solutions at the high Reynolds numbers by a personal computer. The employment of the domain decomposition technology can greatly improve the efficiency of the GBEM and thereby makes it possible for us to get accurate solutions at the high Reynolds number $Re = 5000$ and 7500 by a personal computer. This verifies the great potential of the approached general boundary element method as an efficient numerical method.

Since both the DDM and the parallel computation can improve the efficiency of the GBEM, it is necessary to compare their speedup with equal number of processors and subdomains. Zhao and Liao [7] implemented their GBEM code of parallel computation by the global

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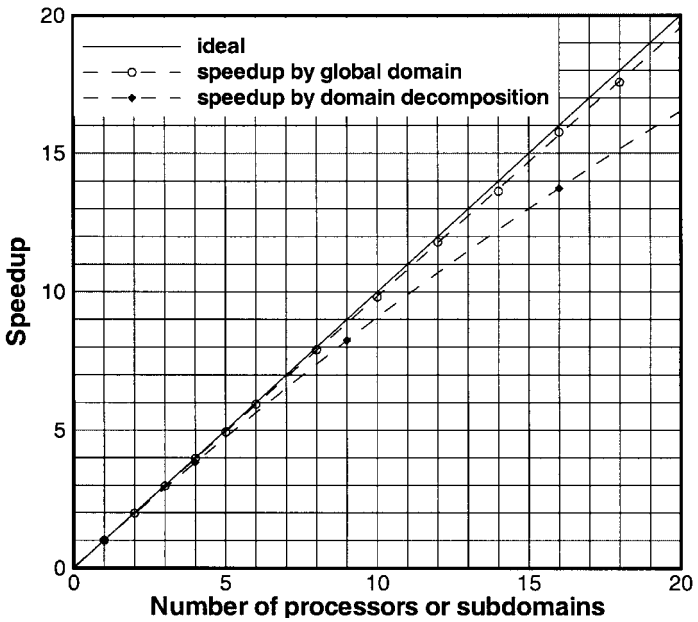

Figure 3. Comparison between the domain decomposition method (without the parallel computation) and Zhao and Liao's [7] parallel computation by the global domain on SGI Onyx 3800 with mesh 120×120.

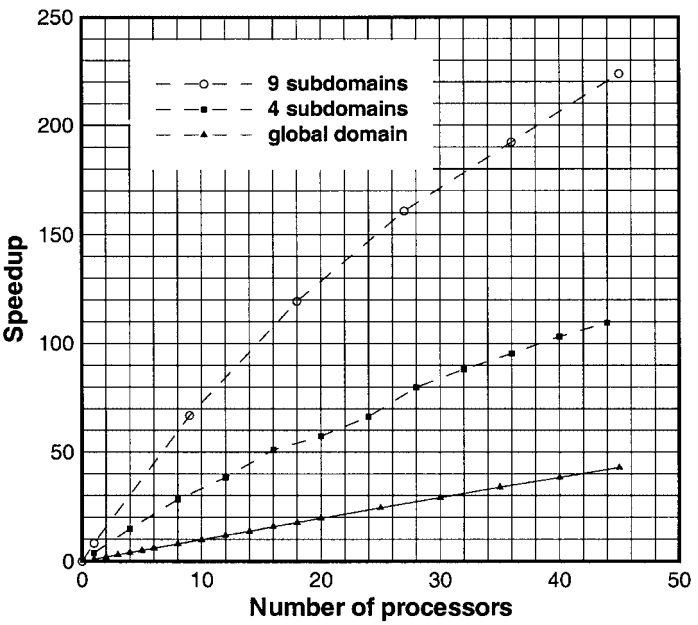

Figure 4. Speedup of the parallel computation by means of the global domain and the domain decomposition on SGI Onyx 3800 with mesh 120×120.

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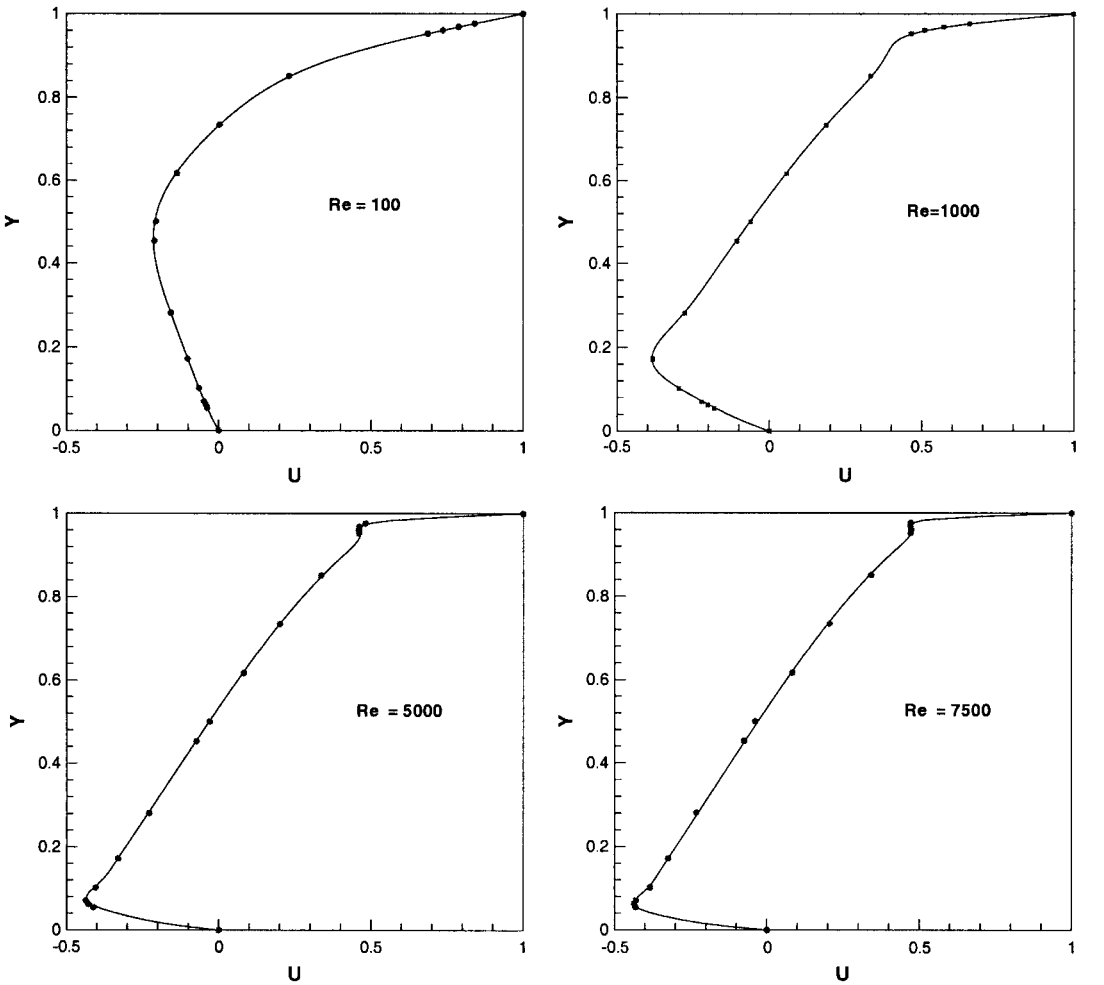

Figure 5. Profiles of velocity u at $x = \frac{1}{2}$ for $Re = 100$, 1000, 5000, and 7500.
Solid line: current result; circle: results given by Ghia *et al.* [30].

domain on SGI Onyx 3800. Their numerical speedup is compared with that given by the DDM on 120×120 mesh grid without the parallel computation, as shown in Figure 3. It seems that the speedup of the parallel computation is only slightly better than that of the DDM, if the number of processors is equal to the number of the subdomains. Therefore, the DDM is a good alternative to the parallel computation for improving the numerical efficiency of the GBEM.

Certainly, the combination of the DDM with the parallel computation can further improve the efficiency of the GBEM. The speedup of our domain decomposition parallel computation is shown in Figure 4. The speedup of the parallel computation by the global domain, i.e. without domain decomposed, is also presented in Figure 4. All of our parallel calculations are

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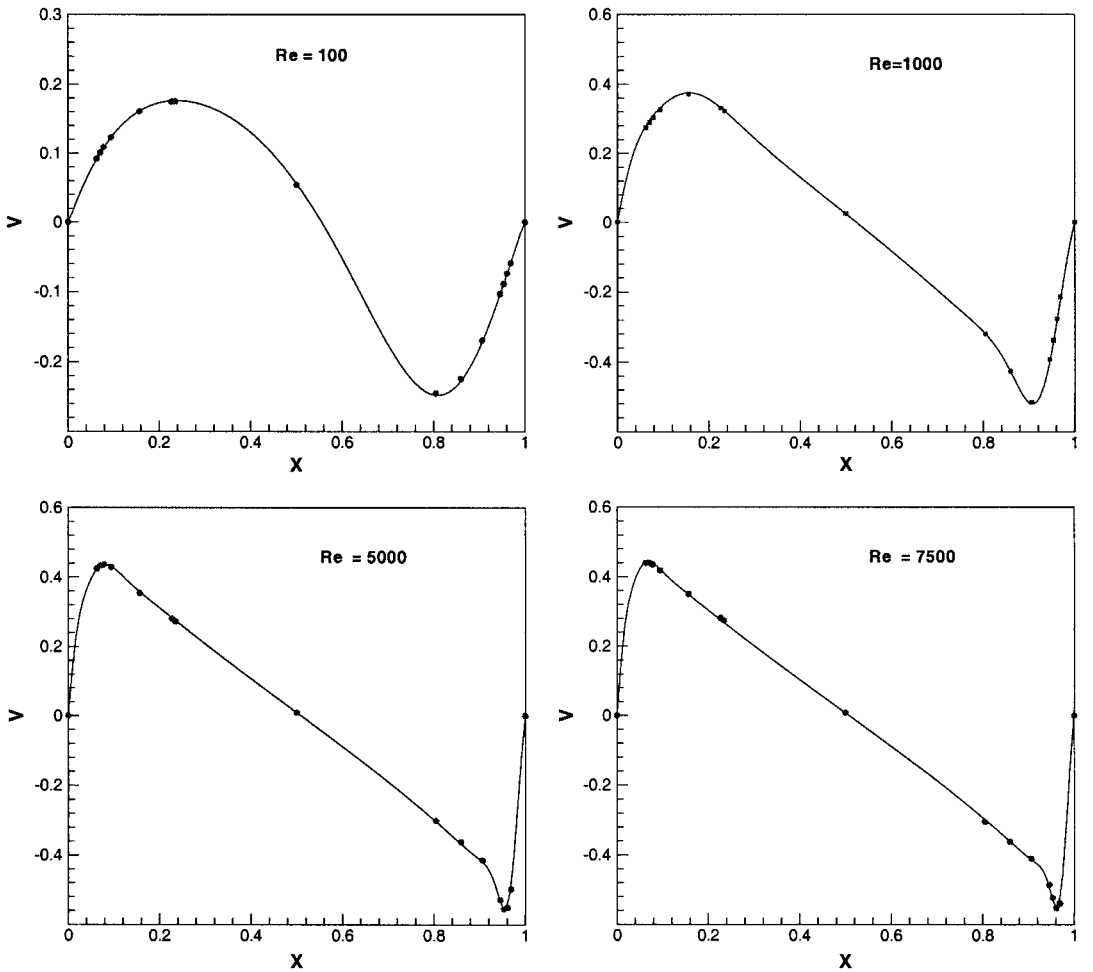

Figure 6. Profiles of velocity v at $y=\frac{1}{2}$ for $Re=100$, 1000, 5000, and 7500. Solid line: current result; circle: results given by Ghia *et al.* [30].

implemented on a high-performance parallel computer SGI Onyx 3800. This figure indicates that, combining the DDM with the parallel computation, the numerical efficiency of the GBEM can be greatly improved.

All of our solutions agree well with those provided by Ghia *et al.* [30] by means of the finite difference method. The numerical results of our approach employing the DDM on 96×96 mesh grid for $Re=100$ and 1000, and on 120×120 mesh grid for $Re=5000$ and 7500, are presented. The velocity profiles of u at $x=\frac{1}{2}$ and v at $y=\frac{1}{2}$, compared with the results given by Ghia *et al.* [30], are as shown in Figures 5 and 6, respectively. The contours of the stream-function are as shown in Figure 7.

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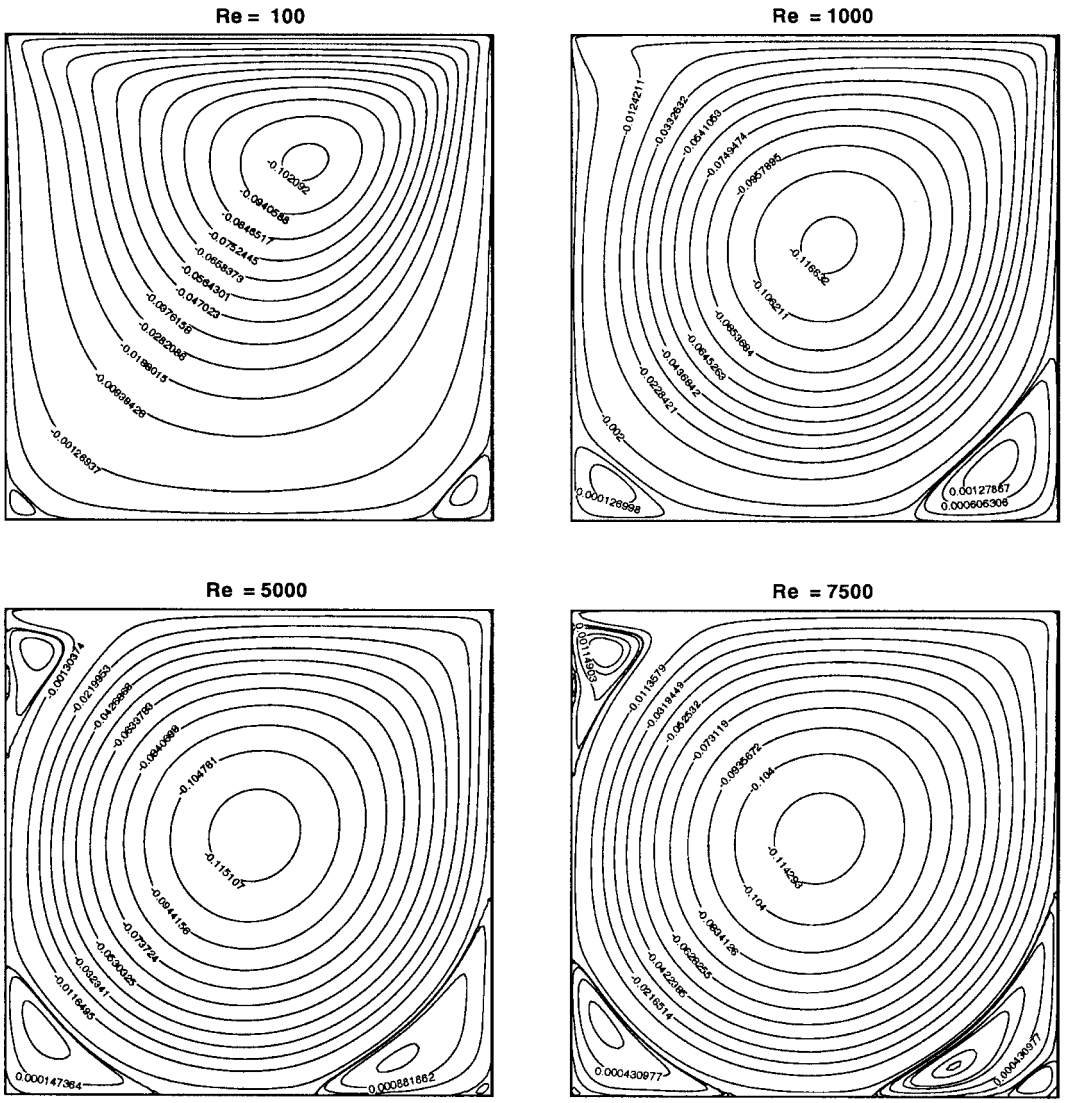

Figure 7. Contour of the stream-function ψ when $Re = 100$, 1000, 5000, and 7500.

6. CONCLUSION

The GBEM has been successfully applied to many non-linear problems [1–7]. However, like the traditional BEM for non-linear problems, the domain integral terms appear, and this greatly decreases the efficiency of the GBEM. To improve the numerical efficiency of the GBEM, Zhao and Liao [7] proposed a kind of parallel computation technique in the GBEM and obtained, for the first time, the solution at $Re = 7500$ for the driven cavity viscous flow by the BEM.

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In this paper, domain decomposition method is applied to further improve the numerical efficiency of the GBEM. Our calculations demonstrate that the domain decomposition method is really a good way to improve the efficiency of the GBEM. Combining the GBEM and the domain decomposition method, we also get the convergent solutions up to $Re = 7500$ in a personal computer. By a high-performance parallel supercomputer, Zhao and Liao [7] obtained the solution up to $Re = 7500$ by the GBEM. We show that, by means of the domain decomposition technique, it is possible to get the convergent solution of the driven cavity viscous flow at $Re = 7500$ even by a personal computer.

The comparison between the speedup of the domain decomposition method and the parallel computation demonstrates that the domain decomposition method is a good alternative to the parallel computation. If computer hardware is limited, we can apply the DDM to gain nearly the same efficiency of the parallel computation.

The domain decomposition method has naturally parallel trends. Our numerical experiments show that the combination of the domain decomposition method with the parallel computation can further greatly improve the numerical efficiency of the GBEM. In this way, the GBEM might become an efficient numerical method for strongly non-linear problems in science and engineering.

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